Complex Numbers

a geometric approach

This lesson offers a short geometric story to introduce the complex numbers before instead of after trigonometry.

The applet shows how to add and multiply points or vectors in the plane)As the easy consequences, the equality of two ways to multiply complex numbers simplifies the derivation of trigonometric identities, gives a simple proof of the cosine, and gives trigonometric formulas for dot- and cross-products. Details follow below. The circular trig part of the site area Maps, Plans, Similarity &Trig, with Complex Numbers continues the development of theory of complex numbers with a discussion of roots of unit and a derivation of of the properties of circular trig functions.

An original contribution below is a high school level proof of the distributive law.

Technical Notes & References

1. In Morris Kline's three-volume work *Mathematical Thought from Ancient to Modern Times,* in volume 2, Chapter 27, the third section called *The Geometrical Representation of Complex Numbers.* This section briefly describes the approach of Caspar Wessel (1745-1818). Part of Wessel's work (translated into English) is reproduced in David Eugene Smith's 1929 work *A Source Book in Mathematics,* Dover 1959 Reprint. In 1976, I saw Richard Feynman in guest lectures at McGill University describe the addition and multiplication of vectors in the plane using parallelograms for addition and the rule *add angles, multiply lengths* for multiplication. Since then I have wondered how to transform his presentation. This essay is the result.

2. **Teachers:** You are welcome to distribute copies or parts of this document in class, as long as the document source (www.whyslopes.com) is acknowledged the copies or in each part distributed. Printable Letter & Legal size versions are available in pdf format. Show this geometric approach to your curriculum or course design and curriculum committees. The arguments below dispersed over a few high school courses may accelerate university and university-oriented mathematics instruction. The presentation below talks about the addition and multiplication of points in the plane. Instead of points, you could introduce rectangular polar coordinate description of vectors or arrows in the plane drawn in standard position, and then talk about coordinate based addition and multiplication of arrows or vectors drawn in standard position.

Step I. Addition and Multiplication

> **Convention:** *Square brackets are used to indicate polar coordinates while round brackets indicate rectangular coordinates.*

Addition (Coordinate Method)

The sum of two points with the rectangular coordinates [*a*,*b*] and [*c*,*d*] is given by

[*a*,*b*] + [*c*,*d*] = [*a*+*c*,*b*+*d*]

Examples:

[2,5]+ [6,2] = [2+6,5+2] = [8,7].

[-1,14]+ [2,8] = [-1+2,14+8] = [1,22].

> **Optional Readings**: For points [*a*,*b*], [*c*,*d*] and [e,f] in the plane, addition is both commutative and associative:
>
> The commutative property
>
> [*a*,*b*]+[*c*,*d*] = [c,d]+[*a*,*b*]
>
> holds as a+ c = c + a and b + d = d + b due to commutative property of the addition of real numbers.
>
> The associative property
>
> ([*a*,*b*]+[*c*,*d*]) +[e,f] = [*a*,*b*]+([*c*,*d*]) +[e,f])
>
> holds as
>
> (a+ c) + e = a + (c + f) and (b +d) + f = b + (d +f)
>
> due to the associative law for real numbers

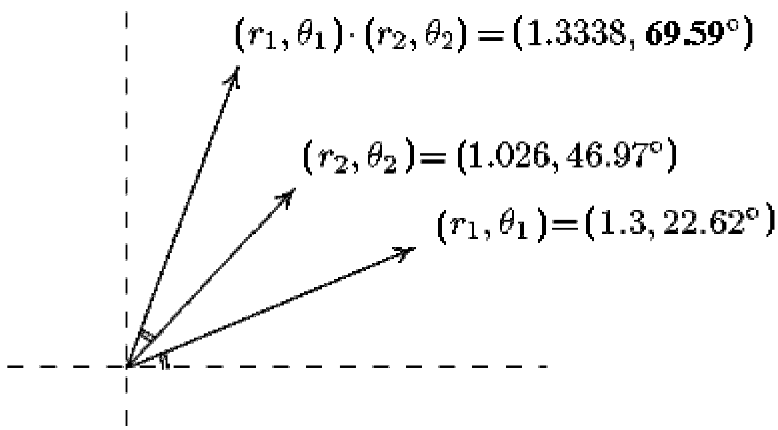
Multiplication with Polar Coordinates

> If point A = [a,b] in rectangular coordinates, then A = (r, θ) in polar coordinates where r is the distance of A to the origin, θ is angle. When A = [a,b] and A = (r, θ), we write [a,b] = (r, θ). Both rectangular and polar coordinates may be use to locate a point in the plane.

The product of two points $[x_1, y_1] = (r_1,\theta_1)$ and $[x_2,y_2] = (r_2,\theta_2)$ in the plane is calculated using the polar coordinates with the *multiply the lengths, add the angles* rule:

$$[x_1, y_1]\cdot[x_2,y_2] = (r_1,\theta_1)\cdot(r_2,\theta_2) = (r_1 r_2,\theta_1+\theta_2)$$

Example. Two arrows are to be multiplied. One has length 1.3 and angle 22.62°; the other factor has length 1.026 and angle 46.97°; and so their product has length 1.3338 = 1.3·1.026 and angle 69.59° = 22.62°+46.97°; and that is it. See the following diagram.



correction: 22.62 + 46.97 = 69.59 not 69.69s

Another Example. The product of the two points (3,80°) and (4, 60°) is

$(3 \cdot 4, 80° + 60°) = (12,140°)$

The product is a third point $[x_3,y_3] = (r_3,\theta_3)$. Its rectangular coordinate $[x_3,y_3]$ are determined by the values of with $r_3 = r_1 r_2$ and $\theta_3 = \theta_1+\theta_2$. Below, we will see how to compute the product using the rectangular coordinates of the factors $[x_1, y_1]$ and $[x_2,y_2]$ directly, without use of polar coordinates.

Optional Readings:

Commutative Law for Products of points in the plane:

$$(r_1,\theta_1)\cdot(r_2,\theta_2) = (r_2,\theta_2)\cdot (r_1,\theta_1)$$

whenever (r_1,θ_1) and (r_2,θ_2) are polar coordinates for a pair of points in the plane. This property follows as the commutative law of addition for real numbers (or angles ≥ 0) implies $\theta_1+\theta_2 = \theta_2+\theta_1$ and the commutative property for products of real numbers (≥ 0) implies $r_1 r_2 = r_2 r_1$,.

Commutative Property for Products of points in the plane:

$$\{(r_1,\theta_1)\cdot(r_2,\theta_2)\}\cdot\ (r_3,\theta_3) = (r_1,\theta_1)\cdot\{(r_2,\theta_2)\cdot\ (r_3,\theta_3)\}$$

whenever (r_1,θ_1), $(r_2,\theta_2)\}$and (r_2,θ_2) are polar coordinates for a pair of points in the plane. This property follows as the associative property of addition for real numbers (or angles ≥ 0) implies $\{\theta_1+\theta_2\}+\theta_3=\theta_1+\{\theta_2+\theta_3\}$ and the commutative property for products of real numbers (≥ 0) implies $\{r_1r_2\}r_3 = r_1\{r_2r_3\}$

Multiplicative Inverse for non-zero points in the plane.

$$(r_1,\theta_1)\cdot(\ [1/r_1]\ ,\ -\theta_1) = (1,0)$$

if (r_1,θ_1) is the polar coordinates of a nonzero point in the plane.

Recapitulation - Summary

The addition of points in the plane is given by means of their rectangular coordinates while multiplication is given in terms of polar coordinates. Below you will see how to multiply points together using rectangular coordinates as well. The equality of different ways to multiply points together leads to many properties of vectors and trigonometry.

Step II. What Are Complex Numbers?

Points in the plane with the operations of addition and multiplication just given are called the complex numbers. The plane with these two operations on its points is called the complex numbers plane, or more briefly the complex numbers.

We will now change to a more standard notation for them. We may and often will write the rectangular coordinates $z = [a,b]$ as $z = a+ib$, We will further call the abscissa a, the real part of the complex number $z = a+ib$. We will also call the ordinate b, the imaginary part of the complex number $z = a+ib$. Note the previous notations [a, b] and (r,θ) will be used for points in the plane in the further discussion of the properties of real and complex numbers. Eventually, the previous notation [a, b] and (r,θ) will be phased out.

Purely Imaginary: We will say that the complex number $z = a+ib$ is purely imaginary when and only when its real part $a = 0$. The angle of a purely imaginary complex number $z = a+ib = 0+ib = (0,b)$ is 90 degrees or 270 degrees (modulo 360 degrees). When $b > 0$, the angle is 90 degrees (modulo 360 degrees). When $b < 0$, the angle is 270 degrees (modulo 360 degrees).

> **Note:** Two quantities x and y are equal modulo a third quantity c, if and only if their difference $x-y = kc$ for some whole number or integer k.

Real: We will also say that $z = a+ib$ is (purely) real when and only when its imaginary part b is zero. The angle of a (purely) real complex number $z = a+ib = a+i0 = (a,0)$ is 0 degrees or 180 degrees (modulo 360 degrees), depending on the sign of the real part a. If $a > 0$, this angle is 0 degrees (modulo 360 degrees) while if $a > 0$, this angle is 180 degrees (modulo 360 degrees).

Real Numbers as Complex Numbers

Each complex number $z = a+i0$ with imaginary part zero gives and is given by a real number a. We will write $z = a$ in this situation, and say that the complex number z is also a real number.

> With this practice, the real numbers can be regarded as a subset of the complex numbers; and the real number line can be identified with the horizontal axis of the plane.

Real and Imaginary Parts

For each point or complex number $z = a+b\ i = (a,b) = [r,\theta]$ in this plane, we say that a is the **real part** of z; that b is the **imaginary part** ***of z;*** that r = |z| is the **magnitude, modulus** or **absolute value** of z (different texts prefer different terms); and that θ is the angle or **argument** of z.

> **Exercise:** Use b = sign(b)|b| to show that bi = b· i where i = [0,1]

Confirmation of The Law of Signs

We identify the real number line with the horizontal axis of the plane. With this identification, observe that positive numbers have angular displacement zero, modulo 360 degrees. Also observe that negative numbers have angular displacement 180 degrees, modulo 360 degrees. The magnitude of a real number is its distance to the origin.

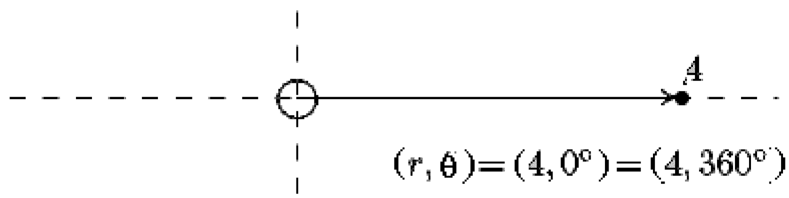
Suppose $z = a+i0$ and $w = c+i0$. We want to compute the product zw with the *multiply the lengths, add the angles* rule. Each factor has length $|a|$ or $|c|$. Each factor has angle 0 or 180 degrees (modulo 360 degrees). The relationships

- $0^\circ = 0^\circ+0^\circ$
- $180^\circ = 0^\circ+180^\circ = 180^\circ+0^\circ$
- $360^\circ = 180^\circ+180^\circ = 0^\circ$ (modulo 360°)

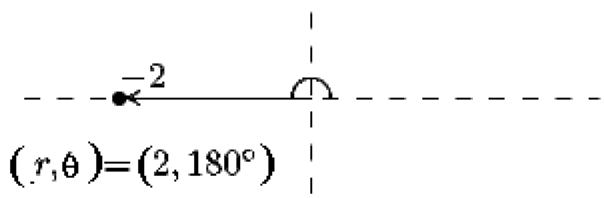
imply the *add the angles, multiply the lengths* rule for the multiplication of complex numbers agrees with the ordinary method for multiplying real numbers and the law of signs. The relationship in particular implies

- $(+1) = (+1)(+1)$ as $0^\circ = 0^\circ+0^\circ$
- $(-1) = (+1)(-1) = (-1)(+1)$ as $180^\circ = 0^\circ+180^\circ = 180^\circ+0^\circ$
- $(-1)(-1) = (+1)$ as $360^\circ = 180^\circ+180^\circ$

Examples and then some further comments may reinforce these ideas. For the first example, the number 4 is now identified with the point $(4,0) = [4,0^\circ] = [4,360^\circ]$. This number or point has distance 4 to the origin and angle of 0°, modulo 360 degrees, with the horizontal axis:



For the second example, the number -2 is identified with the point $[-2,0] = (2,180^\circ)$. See the figure below.



Now multiplying the point $(2,180^\circ)$ by itself leads to the product $(2,180^\circ)^2 = (2^2,180^\circ+180^\circ) = (4,360^\circ) = (4,0^\circ)$. Thus the point on the horizontal axis identified with -2 when squared gives the point identified with +4 indicated above. The 360 degrees in the diagram for the number or point 4 = [4,0] represents the doubling of the angle 180 degrees.

For an example or exercise, compute the pair-wise products of 3=3+0i, 4=4+0i, -3=-3+i0 and -4=-4+0i using the add the angles, multiply the lengths rule.

More Exercises. Compute the following using the multiply the lengths, add the angles rule:

1. $A = (1.5)\cdot(2)$.
2. $B = (1.5)\cdot(-2)$.
3. $C = (-1.5)\cdot(-2)$.
4. $D = (1.5)\cdot(-2)$.
5. $E = (10,45^\circ)\cdot(1/20,15^\circ)$.

Note each factor gives a point or arrow in the coordinate plane.

Stop For A Summary. The polar coordinate definition

$$(r_1,\theta_1)\cdot(r_2,\theta_2) = (r_1 r_2,\theta_1+\theta_2)$$

of the product of two point in the plane, involves the multiplication of lengths (= distances to the origin) and the addition of angles. For points on the horizontal axis, the angles of the factors are zero or 180° (modulo 360°). Computing the angle of the product will involve one of the following expressions:

$$0^\circ+0^\circ = 0^\circ$$

$$0^\circ+180^\circ = 180^\circ$$

$$180^\circ+0^\circ = 180^\circ$$

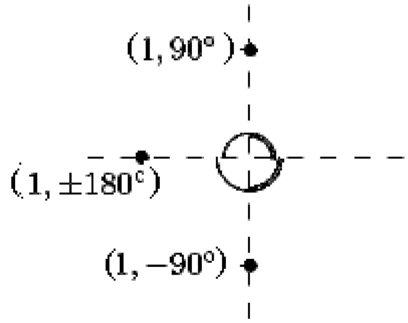
$$180^\circ+180^\circ = 360^\circ$$

since the angle 180 degrees is associated with -1, and the angles 0 and 360 degrees are both associated with the number +1, the polar coordinate definition of multiplication of points in the plane agrees with (or yields) the law of signs for the multiplication of positive and negative numbers.

Square Root of -1

The real number $-1 = -1+0i = (1,180^\circ)$ has angle 180 degrees (mod 360 degrees) and length 1. The purely imaginary number $[0,1] = 0+i1 = (1,90^\circ)$ has angle 90 degrees and length 1. Multiplying this point or number by itself, that is, squaring it, gives the point with length $1 \times 1 = 1$ and angle $90^\circ+90^\circ = 180^\circ$. So the product equals $-1+0i = -1$. We call i, the principal square root of -1.

A second square root of -1 is obtained as follows. The imaginary number $(0,-1) = 0+i(-1) = [1,-90^\circ]$ has angle -90 degrees and length 1. Multiplying this point or number by itself, that is squaring it, gives the point with length 1 times 1 =1 and angle $(-90^\circ)+(-90^\circ) = -180^\circ = 180^\circ$ (mod 360°). So this product equals $-1+0i = -1$ as well.



This provides two square roots of -1 as both $(1,+90^\circ)^2 = (1,+180^\circ) = -1$ and $(1,-90^\circ)^2 = (1,-180^\circ) = -1$.

Square Roots in General

The square root of a positive number or zero are real nonnegative numbers. I assume in the following that you know how to compute these square roots. The square roots of negative numbers and of other arrows or points in the coordinate plane depend on this ability.

Observe that squaring points in the plane doubles their angular displacements and squares their magnitudes (distance to the origin). That is, the *add the angles, multiple the lengths* rule gives

$$[r^{1/2}, \tfrac{1}{2}\theta]\cdot[r^{1/2}, \tfrac{1}{2}\theta] = [r, \theta]$$

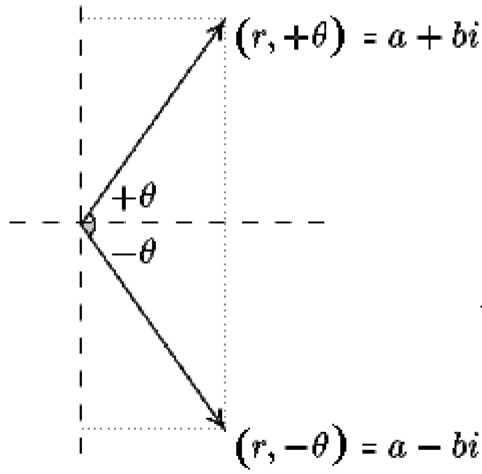
Therefore the arrow $(r^{1/2}, \tfrac{1}{2}\theta)$ when squared (meaning multiplied by itself) yields (r,θ). So it is called a square root of the arrow (r,θ). Another square root is located by the polar coordinates $(r^{1/2}, \tfrac{1}{2}\theta+180^\circ)$ since $(r,\theta) = (r,\theta+360^\circ)$ both locate the same point in the plane. You should consider the special case of positive numbers $z = a+i0 = (a,0^\circ)$ where the angle $\theta = 0$ degrees.

Exercises.

1. Find all the square roots of 4 and -4 and plot them.
2. Find the cube roots of 27 and -27 and plot them in the plane.

Complex Conjugates

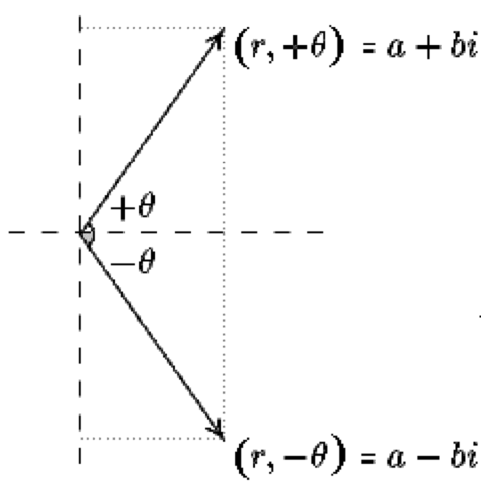
The complex conjugate of a complex number $z = a+b\,i$ with polar coordinates (r, θ) is the complex number $\bar{z} = a-b\,i$ with polar coordinates $(r,-\theta)$. Multiplying a complex number $a+b\,i$ by its conjugate $a-bi$ gives the nonnegative number $r^2 \geq 0$



Conjugates and Multiplicative Inverses (Reciprocals)

Observe that $p = [(a)/(r^2)] - i[(b)/(r^2)] = [1/(r^2)][\overline{(z)}]$ has angle $-\theta$ and length $[1/(r)]$. Here $p = [1/(r^2)][r,-\theta] = (1/r,-\theta.)$ Multiplying number $p = [[1/(r)],-\theta]$ by $z = [r,\theta]$ gives the complex number $[1,0]$ with length 1 and angle 0, that is, the real number 1. And multiplication of any point (c,d) by $1 = [1,0^\circ]$ yields back the point (c,d)

The reciprocal (or multiplicative inverse) of the complex number $z = a+b\,i$ with length $r > 0$ and angle θ is the complex number p with length $1/r$ and angle $-\theta$.



Observe that if $r > 1$ then the length of the reciprocal $[1/(r)] < 1 < r$, that is, the length of the reciprocal is less than 1 and the length of the original number. In contrast, if $0 < r < 1$ then $[1/(r)] > 1 > r$. **Question:** Which of these two cases is represented in the above diagram? What happens in the case $r = 1$?

Some Vocabulary.

For each point or complex number $z = a+b\,i = (a,b) = [r,\theta]$ in this plane, we say that a is the **real part** of z; that b is the **imaginary part of** z; that $r = |z| = \sqrt{(a^2+b^2)}$ is the **magnitude, modulus** or **absolute value** of z (different texts prefer different terms); and that θ is the angle or **argument** of z.

Step III: Arithmetic (Field) Properties of Complex Numbers

Below z, w and v stand denote complex numbers. The following properties are consequences of corresponding properties of real numbers and the rectangular and polar coordinate methods for calculating sums and products of points in the plane.

- **Commutative Property for Addition**: z + w = w+z
- **Commutative Property for multiplication:** zw = wz
- **Additive Identity Exists:** The zero vector 0 = 0 + i 0 has the property 0 + z = z
- **Multiplicative Identity Exist:** The real number 1 = 1 + i 0 has length 1 and angle 0. So it has the property that 1 z = z.
- **Reciprocals (Multiplicative Inverses) Exist for nonzero complex numbers:** If z = (r, −θ) has length r> 0 and angle θ then wz = 1 if w = (1/r, −θ) with length (1/r) and angle −θ.
- **Negatives (Additive Inverses) Exist for all complex numbers:** If z = a + ib = [a, b] then w = (-a) + i(-b) = [-a, -b] has the property that w+z = 0
- **Non Zero Product Property:** If z and w have nonz-zero magnitudes (lengths) r and s then their product has magnitude or length rs > 0. So the product is nonzero

From logic, the equivalent, contrapositive form of the nonzero product law is as follows:

> **Zero Product Law:** If the product wz = 0 for a pair of complex numbers factors z and w then at least one of the factors must be zero.

The **Distributive property says**

$$Z (W + V) = Z W + ZV \quad \text{(left distributive law)}$$
$$(W + V) Z = WZ + V Z \quad \text{(right distributive law)}$$

The appendix below (optional reading) provides a proof. The two laws are equivalent due to the commutative property of multiplication.

Step IV. Multiplication with Real and Imaginary Parts

The distributive law applied twice implies

$$(a+ib)(c+id) = (ac-bd)+i(bc+ad)$$

for the product of two points in the plane in terms of their rectangular coordinates, alias real and imaginary parts.

Proof:

(a + bi) (c+ di) = a(c+di) + bi (c+ di)
(by first use of distributive law)

= ac+ a(di) + (bi)c+ (bi)(di)
(by second use of distrributivelaw)

= ac + i ad + i bc + (-1) bd
(by associative and commutative law for products)

= ac + (-1) bd + i ad + i bc
(by associative and commutative laws for sums)

= 1 (ac + (-1) bd) + i (ad + bc)
(by the distributive law in reverse)

= [ac + (-1) bd , ad + bc]

The foregoing gives a second way to multiply complex numbers together using their real and imaginary parts

(a + bi) (c+ di) = (ac - bd) + i (ad + bc)

or equivalently, with or rectangular coordinates notation,

[a,b] [c,d] = [ac -bd, ad+ bc]

The latter formulas often the starting point for the definition of products of complex numbers before the introduction of complex number notation in the plane.

Exercise: Use b = sign(b)|b| to show that bi = b· i where i = [0,1]

Step V: Consequences of Two Ways to Compute Products

Mathematicians, engineers and physicist know well how the properties of complex numbers and the function $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta) = e^{i\theta}$ simplifies the development of trigonometry identities. The following easy consequences are likely to be less well-known.

A. Trig Identities Simplified

From trigonometry, recall the unit circle definitions of the sine and cosine functions. Let

$$e^{i\theta} = \text{cis}(\theta) = \cos(\theta)+i\sin(\theta) = \exp(i\theta) = [\cos(\theta), \sin(\theta)] = (1, \theta)$$

Now $(1,A)\cdot(1B) = (1, A+B)$. Therefore property

$$\text{cis}(A)\cdot\text{cis}(B) = \text{cis}(A+B)$$

follows from the above *add the angles, multiply the lengths* definition of complex multiplication.

From $\text{cis}(A) = \cos(A)+i\sin(A) = a + bi$, and $\text{cis}(A) = \cos(B)+i\sin(B) = c + id$

$$\text{cis}(A+B) = \text{cis}(A)\cdot\text{cis}(B) = (a+ib)(c+id) = (ac-bd)+i(bc+ad) = \{ \cos(A)\cos(B) -\sin(A)\sin(B) \} + i\{\sin(A)\cos(B)+ \cos(A)\sin(B)\}$$

But $\cos(A+) = \cos(A+B) + i \sin(A+B)$ as well. Therefore in rectangular coordinates

$$[\cos(A+B), \sin(A+B)] = [\cos(A)\cos(B) -\sin(A)\sin(B), \sin(A)\cos(B)+ \cos(A)\sin(B)]$$

Equality requires the angle sum formulas to hold.

$$\cos(A+B) = \cos(A)\cos(B) -\sin(A)\sin(B)$$

$$\sin(A+B) = \sin(A)\cos(B)+ \cos(A)\sin(B)$$

The verification or derivation of trig identities can be reduced to algebraic manipulations involving

$$\text{cis}(\theta) = \cos(\theta) + i\sin(\theta) = \exp(i\,\theta)$$

B. Trigonometric Formulas for Dot and Cross Products

Suppose $[x_1,y_1] = [r_1 \cos(\theta_1), r_1 \sin(\theta_1)]$ and $[x_2,y_2] = [r_2 \cos(\theta_2), r_2 \sin(\theta_2)]$ are points in the plane. Then their dot product

$$[x_1,y_1]\bullet[x_2,y_2] = x_1x_2+y_1y_2 \quad \text{(dot product definition)}$$

and their cross product

$$[x_1,y_1]\bullet[x_2,y_2] = \; x_1y_2 - y_1x_2 \quad \text{(cross product definition)}$$

may be expressed in terms of trigonometric functions and the angles between the two points, or more precisely their position vectors. See below.

Details

To obtain the geometric interpretation, observe the polar and rectangular ways to multiply the first point by the complex complex conjugate of the second, when both are viewed as complex numbers: That is,

$$[x_1,y_1][x_2, -y_2] = (r_1 , \theta_1)(r_2, -\theta_2)$$

From the equality of two different ways to multiply points in the plane, observe

$$[x_1x_2+y_1y_2 , x_1y_2 - y_1x_2] = (r_1r_2,\theta_1- \theta_2)$$

but

$$(r_1r_2,\theta_1- \theta_2) = [r_1r_2 \; \cos(\theta_1- \theta_2), r_1r_2 \; \sin(\theta_1- \theta_2)]$$

Therefore comparison (equality) of real and imaginary parts yields:

$$x_1x_2+y_1y_2 = r_1r_2 \; \cos(\theta)$$

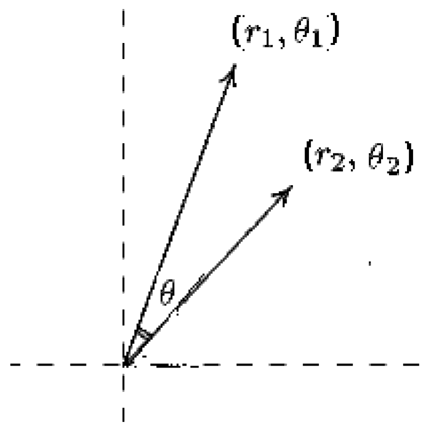
and

$$x_1y_2 - y_1x_2 = r_1r_2 \; \sin(\theta)$$

where

$$\theta = \theta_1- \theta_2$$

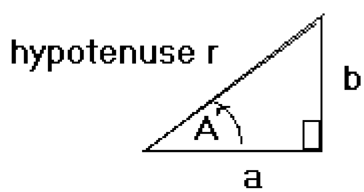
is the angle between the two points $[x_1,y_1] = (r_1 , \theta_1)$ and $[x_2,y_2] = (r_2 , \theta_2)$



C. Pythagorean Theorem

In the development of complex numbers, we may geometrically imply that the product (r,0)·[a,b] = [ra, rb] = r·[a,b] is given by a scalar multiplication, and that the midpoint of the line segment from [a,b] to [c, d] is ½·[a+c,b+d] via similarity and triangle isometry (congruency) arguments. With that the properties of complex numbers employed are not derived from the Pythagorean theorem.

> Theorem: If a right triangle has a hypotenuse of length r and other two sides of lengths *a* and *b*,

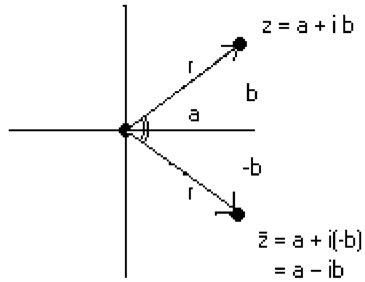


then

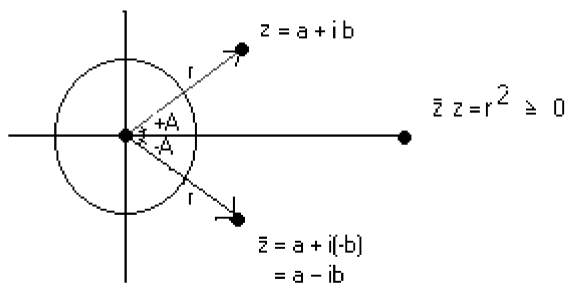
$$a^2+b^2 = r^2$$

Complex Number Proof

Let z = a + i b = (r, A) be a point in the first quadrant. The triangle with vertices 0, a, a+ib is congruent or isometric to the given right triangle



Multiplying a vector a +i b with angle A and length r by its complex conjugate a -ib gives a complex number with angle A + (-A) = 0 and length r^2 units according to the *add the angles, multiply the lengths* polar coordinate, multiplication rule. The product has value $r^2 \geq 0$ as shown below.



This gives

$$r^2 = (a+ib)\ (a-ib)$$

But the previous formulas for expressing products in terms of their real and imaginary parts,

$$r^2 = a^2 + b^2 + i(-ba+ab)$$

as the polar and rectangular methods for computing a product give the same result.

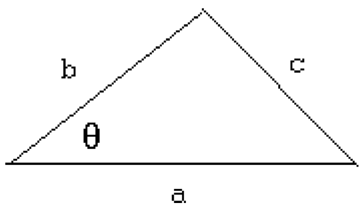
This implies

$$r^2 = a^2 + b^2$$

Small Print: There are over 100 different proofs of the Pythagorean theorem. Using the Chinese square dissection method to imply the Pythagorean theorem, or another method to imply the Pythagorean theorem and then use the latter instead of triangle similarity and isometric arguments to imply that the product (r,0)·[a,b] = [ra, rb] = r·[a,b] is given by a scalar multiplication, and that the midpoint of the line segment from [a,b] to [c, d] is ½·[a+c,b+d] gives an easier route for the exposition of complex numbers. Whence the above arguments simplify shows the Pythagorean theorem is consistent with the geometric properties of complex numbers.

D. The Cosine Law

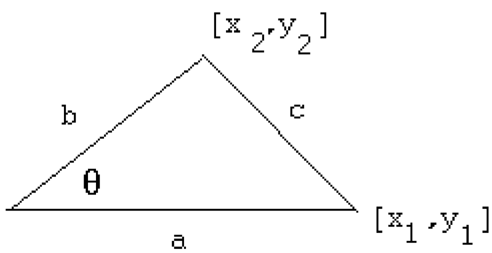
For a triangle, with sides of length a, b and c, and angle θ opposite the side of length c,



the cosine law say $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.

Proof of the Cosine Law

Introduce a coordinate system so that the origin z = 0 is at the angle and the ends of the adjacent sides have coordinates $[x_1,y_1]$ and $[x_2,y_2]$. Then the following diagram and computation of the dot product implies



and computation of the dot product implies

$$x_1x_2+y_1y_2 = r_1r_2\ \cos(\theta) = ab\cos(\theta)$$

But

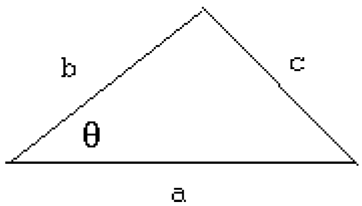
$$c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1^2 - 2x_1x_2 + x_2^2) + (y_1^2 - 2y_1y_2 + y_2^2)$$

$$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2)$$

$$= a^2 + b^2 - 2ab\cos(\theta)$$

Pythagorean Theorem Converse:

If the sides of a triangle have length a, b and c with $c^2 = a^2 + b^2$ then the triangle has a right angle opposite the side of length c.



Proof of Converse:

When $0 < \theta < 180$,

$\cos(\theta) = 0$ when and only when $\theta = 90$ degrees

Therefore $c^2 = a^2 + b^2 - 2ab\cos(\theta) = a^2 + b^2$ with $a > 0$, $b > 0$ and $0 < \theta < 180$ implies $\cos(\theta) = 0$ and hence $\theta = 90$ degrees. So the triangle is a right triangle.

~~E. Trig Identities Simplified - Easy Consequence A, continued~~

Formulas for cos(2A) and sin (2A)

From cis(2A) = cis(A)*cis(A) we found

$\cos(2A) + i\sin(2A) = (\cos(A) + i\sin(A))(\cos(A) + i\sin(A))$

$= \cos^2(A) - \sin^2(A) + i\, 2\cos(A)\sin(A)$

Therefore comparison of rectangular coordinates/components (or real and imaginary parts) yields the double angle formulas

$\cos(2A) = \cos^2(A) - \sin^2(A)$

$\sin(2A) = 2\, i \cos(A)\sin(A)$

Formulas for cos(nA) and sin (nA), the case n =3.

Observe

$$\exp(i3A) = \cos(3A) + i\sin(3A) = \exp(iA)\exp(iA)\exp(iA)$$

Therefore

$\exp(i3A) = \{\cos(A) + i\sin(A)\}^3$

$$= .\cos^3(A) + 3\cos^2(A)\, i \sin(A) + 3 \cos(A) \{i \sin(A)\}^2 + (i \sin(A))^3$$

$$= .\cos^3(A) - 3 \cos(A) \sin^2(A) + i\{3\cos^2(A) \sin(A) - \sin^3(A))$$

Equality of corresponding real and imaginary parts gives

$$\sin (3A) = 3\cos^2(A) \sin(A) - \sin^3(A) = 3(1- \sin^2(A))\sin(A) - \sin^3(A)$$

$$= 3 \sin(A) - 4 \sin^3(A)$$

$$\cos(3A) = .\cos^3(A) - 3 \cos(A) \sin^2(A) = \cos^3(A) - 3 \cos(A) (1-\cos^2(A))$$

$$= 4 \cos^3(A) - 3 \cos(A)$$

Exponential or cis Expressions for Trig Functions

From the two equations

$$\exp(iA) = [\cos(A),\sin(A)] = \cos(A) + i \sin(A)$$

$$\exp(-iA) = [\cos(-A),\sin(-A)] = \cos(A) - i \sin(A)$$

we see

$$\exp(iA) + \exp(-iA) = 2 \cos(A)$$

and

$$\exp(iA) - \exp(-iA) = 2i \sin(A)$$

Therefore

$$(1/2)[\exp(iA) + \exp(-iA)] = \cos(A)$$

and

$$(1/2i) [\exp(iA) - \exp(-iA)] = \sin(A)$$

Expressions involving exp(iA) and exp(-iA) now follow from:

$$\tan(A) = {}^{\sin(A)}/_{\cos(A)}$$

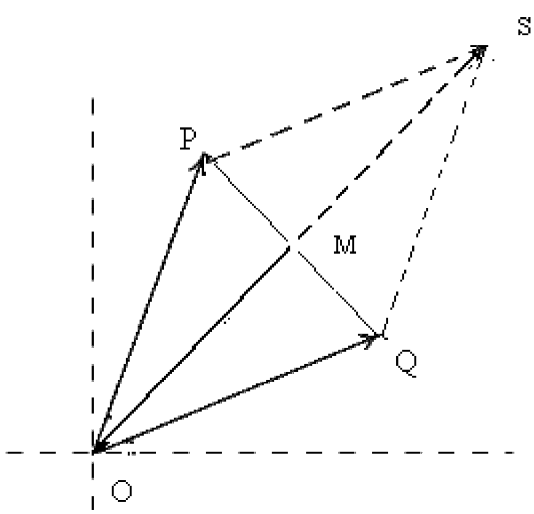
So all trig functions may be expressed in terms of exp(iA) and exp(-iA). The substitution of exp(iA) and exp(-iA) expressions for trig functions turns the proof or derivation of trig identities into simple algebraic exercises involving complex numbers and the add the angles rule for multiplication of exp(iA) with exp(iB). While highschool students may be taken through the exercises of proving trig identities before meeting complex numbers, the above quick explanation of complex numbers and its links to trig imply a quick route through high school mathematics courses on algebra and trig. In retrospect, the presentation of trig in highschool has been harder than need-be.

End Notes

Trig course today could cover the above material, show how most trig identities follow from calculations with complex numbers, and give applications of trigonometry to distance calculations based on the similarity of right triangles and the values of trigonometric functions. A course on trig and complex numbers could explore more analytic geometry, show how to compute powers and roots for positive real numbers using the natural logarithm (defined for positive numbers) and exponential functions (defined for real numbers), and then extend these definitions to give definitions of powers and roots for complex numbers, including negative real numbers. Calculations of roots of unity would further tie trigonometry and complex numbers together. More motivation and more applications of trig, tied to right triangles and complex numbers, come from engineering and physics. The description of electric currents and devices depends on sine and cosines, or more simply, if you know complex numbers and exp(iA) = exp(iA). The latter functions also appear in the theoretical treatment of statistics and of higher mathematics (analysis).

Appendix: Proof of the Distributive Law

Let P and Q be points in the plane. The midpoint of P and Q is M = ½ ·(P+Q) = ½· S where S is the sum of the two points



O = [0,0]	P = [a,b] = (r_1, θ_1)	Q = [c, d] =
	S = [a, b] + [c, d]	M = $\frac{1}{2}$ {[a,b

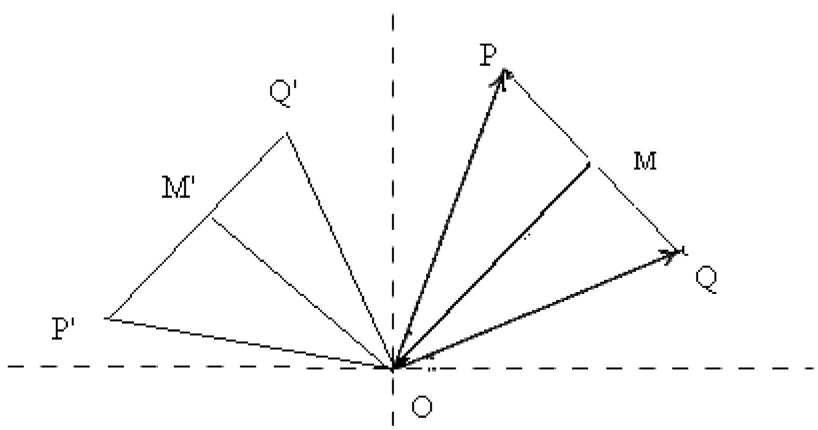
Figure 1: Two Points P and Q in the Plane and the origin determine a triangle

POQ

Multiplication by $(r,\theta) = (1,\theta)\cdot(r,0)$ is the same as multiplication by a unit length rotation $(1,\theta)$ and a zero angle $(r,0)$ point. But multiplying by a zero angle point is equivalent to the scalar multiplication $r\cdot[a,b] = [ra,rb]$, and that is distributive.

In general, save for a collinear case to be treated separately, the two points $P = [a,b] = (r_1,\theta_1)$ and $Q = [c,d] = (r_2,\theta_2)$ along with the origin [0,0] are the vertices of a triangle The midpoint M of the side opposite the origin has rectangular coordinates ½ { $[a,b] + [c,d]$ }= ½ { P + Q }= ½ S. But multiplication by $(1,\theta)$ rotations the vertices into $P' = (1,\theta)\cdot[a,b] = (r_1,\theta + \theta_1)$, $(1,\theta)\cdot[c,d] = (r_2,\theta+\theta_2)$, and $Q' = (1,\theta)\cdot[0,0] = [0,0]$. The latter provide vertices of a second triangle.

Next we show that rotation by angle θ Distributes over Midpoint Calculation. Details follow.



Multiplication by $(1,\theta)$ rotates P into $P' = (r_1, \theta + \theta_1)$ and Q into $Q' = (r_2, \theta + \theta_2)$

The angles ∡POQ and ∡P'OQ' are both equal to $\theta_1 - \theta_2$
while the adjacent sides OP and OQ to ∡POQ have the same lengths r_1 and r_2
respectively, as the adjacent sides OP' and OQ' to ∡P'OQ'
Therefore the triangles △POQ and △P'OQ' are isometric by the SAS criteria.

By isometry of triangles $\triangle$POQ and $\triangle$P'OQ', the angles $\angle$M'Q'O and $\angle$MQO are equal. By isometry, the side PQ and P'Q' have the same length. Let M' be the midpoint of P'Q'. It distance to Q' is half that of P'Q'. Let M be the midpoint of PQ. It distance to Q is half that of PQ. Therefore line segments Q'M' and QM adjacent to angles $\angle$M'Q'O and $\angle$MQO have the same length. while OQ and OQ' have the same length r_2.
Therefore the triangles $\triangle$M'Q'O and $\triangle$MQO are isometric by the SAS criteria.

The latter isometry implies OM' and OM have the same length b. It also implies angles $\angle$M'OO' and $\angle$MOQ have the same value $\Delta\theta$ Therefore in polar coordinates
$M = (b, \Delta\theta + \theta_2)$ while $M' = (b, \Delta\theta + \theta + \theta_2) = (1,\theta)\cdot(b, \Delta\theta + \theta_2) = (1,\theta)\cdot M$

Thus $M' = (1,\theta)\cdot M$ But $M = \frac{1}{2}\{P+Q\}$ and $M' = \frac{1}{2}\{P' + Q'\} = \frac{1}{2}\{(1,\theta)\cdot P + (1,\theta)\cdot Q\}$

Therefore $(1,\theta)\cdot\frac{1}{2}\{P+Q\} = \frac{1}{2}\{(1,\theta)\cdot P + (1,\theta)\cdot Q\}$

Apart from the factor $\frac{1}{2}$, the latter is the distributive law for multiplication by $(1,\theta)$

Scalar multiplication by 2 is equivalent to multiplication by the zero angle point (2,0) and multiplication commutes. A scalar multiplication by 2 now implies the distributive law for multiplication by $(1\ \theta)$.

$$P' + Q' = 2M' = 2(1,\theta)\cdot M = (1,\theta)\cdot 2M = (1,\theta)(P + Q)$$

Therefore

$$(1,\theta)\cdot P + (1,\theta)\cdot Q = (1,\theta)\cdot(P + Q)$$

Multiplication by $(r,\theta) = (1,\theta)\cdot(r,0)$ being equivalent to two successive distributive operations is also distributive.

The case where the origin O =[0,0], and points P and Q are collinear follow from similar or easier arguments in the twosubcases where P and Q are on (i) the same side and (ii) opposite sides of the origin.

Variation of Proof: If we were to take P = [2a,2b] $=(2r_1,\theta_1)$ and Q = [2c, 2d] $=(2r_2,\theta_2)$ above then the midpoint M = ½ · {P + Q} = [a,b] + [c,d]. Further the conclusion $(1,\theta)\cdot M = M'$ would follow as above. That implies

$(1,\theta)\cdot\{[a,b] + [c,d]\} = (1,\theta)\cdot M = M'$

$= ½ \cdot \{P' + Q'\}$
$= ½ \cdot \{ (1, \theta)\cdot 2 [a,b] + (1, \theta)\cdot 2[c,d]\}$
$= (1, \theta)\cdot[a,b] + (1, \theta)\cdot [c,d]\}$

Remark for Students of Differential Geometry: The observation that the sum of two points [a,b] and [c,d] in the plane is the midpoint of the line segment joining [2a,2b]=$(2r_1,\theta_1)$ and [2c, 2d] =$(2r_2,\theta_2)$ provides a geometric alternative to the parallelogram method for obtaining the sum. On Riemann surface, choose a point O to serve as the origin of a geodesic based polar coordinate system (r, θ). Locally, let the (symmetric) sum S of two points (r_1,θ_1) and (r_2,θ_2) be given by the midpoint of the geodesic joining $(2r_1,\theta_1)$ and $(2r_2,\theta_2)$. And in that polar coordinate system, for each positive number K, the K-product is given by $(r_1,\theta_1)\cdot(r_2,\theta_2) = (Kr_1r_2,\theta_1+\theta_2)$ for some Thus two operations may be locally defined in a neighborhood of point on a Riemann surface. If the polar disk $r_1 \leq 1/K$ lies in that neighborhood, it is closed under the K-product operation. The circle of radius 1/K is invariant When the Riemann surface is rotational symmetric about the origin of the polar system, as in the case of the sphere or plane, rotations distribute over addition but dilations $c\,(r, \theta) = (cr, \theta)$ on the sphere do not.

Question: For the addition of points a and c on the real number line, let a + c be the midpoint for the line segment with vertices 2a and 2c. Clearly a + c = c + a. Is it is possible for that introduction of addition of points on the real number line to aid the development of arithmetic with whole numbers and fractions or integers and rational numbers?